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Fuzzy Funnels: Non-abelian Brane Intersections ¹

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ABSTRACT

We discuss dual formulations of D-brane intersections. The duality is between world volume field theories of different dimensionalities which both describe the same D-brane configuration but are valid in complementary regions of parameter space. We discuss the duality in terms of bion configurations involving D-strings orthogonally intersecting both D3-branes and D5-branes.

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1 Introduction

In string theory we have become accustomed to the notion that many physical systems possess several equivalent or rather dual mathematical descriptions which may be distinguished by their usefulness in different regions of parameter space. The S-,T-,U-dualities and AdS/CFT are perhaps the most well known and useful of these relations. Of these AdS/CFT [1] is a remarkable example of a duality relating theories in different space-time dimensions. In this case the duality is holographic in the sense that a theory of gravity living in an AdS space is completely described by a quantum field theory living on the “boundary” of the AdS space. Holography however need not be a pre-requisite for such relationships between theories in different dimensions. Here we will discuss two simpler examples of systems which have alternate descriptions as field theories in different dimensions. First we consider the well known example of D1-branes orthogonally intersecting a D3-brane. In the low-energy world volume theory of the D3-brane this is described as a magnetic monopole plus a non-trivial profile for one of the transverse scalar fields in the $(3 + 1)$ -dimensional Born-Infeld action[2]. If one is interested in the dynamics of this system far away from the core of the monopole then one simply expands the Born-Infeld action around the monopole-scalar background. If on the other hand one wishes to probe this system near the point at which the D1-branes are attached to the D3-brane, *i.e.*, near the core of the magnetic monopole, the low energy theory on the D3-brane is no longer valid as the fields are rapidly varying on the string scale at this point. Fortunately this D-brane intersection has another description in terms of the $(1 + 1)$ -dimensional theory living on the D1-branes[3, 4]. Here the D1-brane system is found to have non-commutative solutions in which three of the transverse scalar fields open up into a fuzzy funnel (so called because the cross sections are fuzzy two-spheres) as one moves along the D1-branes. This funnel turns out to be a source for the correct Ramond-Ramond (RR) field to be identified with an orthogonal D3-brane[4]. Since the $(1 + 1)$ -dimensional Born-Infeld action can be trusted when the radius of the funnel is small, this formulation provides an accurate description of the physics near the point of attachment, or rather at the monopole core. Thus the $(3 + 1)$ -dimensional theory on the D3-branes and the $(1 + 1)$ -dimensional theory on the D-strings provide dual frameworks with which to describe the $D3 \perp D1$ intersection. In either case, the low energy world volume theory cannot provide a complete account of this system, but what is remarkable is the degree to which the two descriptions agree in the limit of large magnetic charge[4].

Another example of this type of duality is provided by the orthogonal intersection of D1-branes and D5-branes recently examined in ref. [5]. This system is very similar to the $D3 \perp D1$ -system in that the theory on the world volume of the D5-branes involves a non-trivial gauge field configuration as well as a scalar field profile giving rise to a bion spike configuration. Again, near the core of the spike the $(5 + 1)$ -dimensional Born-Infeld action is no longer trustworthy and we must revert to the dual description provided by the D1-branes. As in the D3-brane case an orthogonal D5-brane can be realised as a non-commutative funnel involving five of the transverse scalars with a cross section given by fuzzy four-spheres. The configuration carries the correct charge to be identified with a D5-brane and as before provides an accurate description of the physics near the point at which the D1-branes are attached to the the D5-branes.

The remainder of this note is organized as follows. In section 2, we review the dual formulations of the D3⊥D1-intersections as presented in ref. [4]. In section 3, we provide a similar discussion of the the D5⊥D1 system, as given in ref. [5]. In section 4, we conclude with a discussion of further applications of these dualities and some open questions.

2 Dual Formulations of D3⊥D1 Intersections

It was originally recognized in ref. [2] that the world volume theory of a D-brane supports “solitonic” configurations corresponding lower-dimensional branes protruding from the original D-brane. In particular, in the context of D3-branes, BPS monopoles correspond to an orthogonal D-string. This system has been extensively studied in the literature[6, 7, 8] and we review some of the salient results here.

The low energy dynamics of a (single) D3-brane in a Minkowski space background is described by the Born-Infeld action[9],

$$S = \int dt \mathcal{L} = -T_3 \int d^4\sigma \sqrt{-\det(\eta_{ab} + \lambda^2 \partial_a \phi^i \partial_b \phi^i + \lambda F_{ab})}, \quad (1)$$

where $\lambda \equiv 2\pi\alpha' = 2\pi\ell_s^2$. Here we have implicitly used static gauge, with σ^a ($a = 0, \dots, 3$) denoting the world-volume coordinates while ϕ^i ($i = 4, \dots, 9$) are the scalars describing transverse fluctuations of the brane. F_{ab} is the field strength of the $U(1)$ gauge field on the brane.

General considerations of intersecting branes[10] indicate that a D-string ending on a D3-brane should act as a source of the first Chern class of the world-volume gauge field. Hence, as mentioned above, D-strings appear as BPS magnetic monopoles within the three-brane world volume[2]. The solution involves exciting one of the scalar fields, say $\phi \equiv \phi^9$, as well as a magnetic field $B^r = \frac{1}{2}\epsilon^{rst}F_{st}$ ($r, s, t = 1, \dots, 3$). One can show that it is consistent with the equations of motion to set all the other fields equal to zero. For static configurations, we then evaluate the energy,

$$\begin{aligned} E = -\mathcal{L} &= T_3 \int d^3\sigma \sqrt{\lambda^2 |\vec{\nabla}\phi \mp \vec{B}|^2 + (1 \pm \lambda^2 \vec{B} \cdot \vec{\nabla}\phi)^2} \\ &\geq T_3 \int d^3\sigma \left(1 \pm \lambda^2 \vec{B} \cdot \vec{\nabla}\phi\right). \end{aligned} \quad (2)$$

The first term in this lower bound is simply the energy of the D3-brane. The second term is a total derivative: $\vec{B} \cdot \vec{\nabla}\phi = \vec{\nabla} \cdot (\vec{B}\phi)$ using the Bianchi identity $\vec{\nabla} \cdot \vec{B} = 0$. Hence this term is topological, depending only on the boundary values of the fields specified at infinity and near singular points (after introducing a cutoff, such as near $r = 0$ below). Therefore, the last line of eq. (2) provides a true minimum of the energy for a given set of boundary conditions[11].

The lower bound in eq. (2) is achieved when

$$\vec{\nabla}\phi = \pm \vec{B}, \quad (3)$$

which coincides with the BPS condition for the magnetic monopoles[2]. Further, using the Bianchi identity, eq. (3) implies $\vec{\nabla}^2\phi = 0$ and so we are looking for harmonic functions on the D3-brane. The simplest solution, corresponding to the “bion spike,” is given by

$$\phi = \frac{N}{2r} , \quad \vec{B} = \mp \frac{N}{2r^3} \vec{r} , \quad (4)$$

where $r^2 = (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2$, and N is an integer due to quantization of the magnetic charge. The energy of this configuration is readily computed to be

$$E = T_3 \int d^3\sigma + NT_1 \int_0^\infty d(\lambda\phi), \quad (5)$$

where $T_1 = (2\pi\ell_s)^2 T_3$. Here we note that $\lambda\phi$ is the physical distance in the transverse direction. Hence we have recovered the energy expected for a BPS configuration consisting of N semi-infinite D-strings ending on an orthogonal D3-brane. Other considerations, such as the effective charge distribution and fluctuations on the strings[6, 7], further confirm that the bion spike describes the D3 \perp D1-system to a good approximation.

The D3 \perp D1-system has a dual description in the D-string theory, which has also received a great deal of attention in the literature[3, 11, 12]. As emphasized in ref. [4], interpreting the D-string world volume solutions in terms of noncommutative geometry completes the correspondence to the results in D3-brane theory. The low energy dynamics of N D-strings in a flat background is well described by the non-abelian Born-Infeld action[13, 14]:

$$S = -T_1 \int d^2\sigma \text{STr} \sqrt{-\det(\eta_{ab} + \lambda^2 \partial_a \Phi^i Q_{ij}^{-1} \partial_b \Phi^j)} \det(Q^{ij}) , \quad (6)$$

where $Q^{ij} = \delta^{ij} + i\lambda[\Phi^i, \Phi^j]$. Again we are assuming static gauge where the two worldsheet coordinates are identified with $\tau = x^0$ and $\sigma = x^9$. The transverse scalars Φ^i ($i = 1, \dots, 8$) are now $N \times N$ matrices transforming in the adjoint representation of the $U(N)$ worldsheet gauge symmetry. The symmetrized trace prescription[15], denoted by STr , requires that we symmetrize over all permutations of $\partial_a \Phi^i$ and $[\Phi^i, \Phi^j]$ within the gauge trace upon expanding the square root.

To find static solutions describing the D3 \perp D1-system, we allow three of the transverse scalars, Φ^i ($i = 1, 2, 3$), to depend on the spatial coordinate σ . Evaluating the determinants in eq. (6) the energy becomes

$$\begin{aligned} E &= T_1 \int d\sigma \text{STr} \sqrt{\lambda^2 (\partial_\sigma \Phi^i \mp \frac{i}{2} \epsilon^{ijk} [\Phi^j, \Phi^k])^2 + (1 \pm \frac{i}{2} \lambda^2 \epsilon^{ijk} \partial_\sigma \Phi^i [\Phi^j, \Phi^k])^2} \\ &\geq T_1 \int d\sigma \text{STr} \left(1 \pm \frac{i}{2} \lambda^2 \epsilon^{ijk} \partial_\sigma \Phi^i [\Phi^j, \Phi^k] \right) \\ &= NT_1 \int d\sigma \pm \frac{i}{3} \lambda^2 T_1 \int d\sigma \partial_\sigma \text{Tr} (\epsilon^{ijk} \Phi^i \Phi^j \Phi^k) . \end{aligned} \quad (7)$$

Here the lower bound is again the sum of a trivial term (the energy of the N D-strings) and a topological term. The minimum energy condition is

$$\partial_\sigma \Phi^i = \pm \frac{i}{2} \epsilon^{ijk} [\Phi^j, \Phi^k], \quad (8)$$

which can be identified as the Nahm equations[3, 16]. The solution dual to a single-centered BPS monopole is given by

$$\Phi^i(\sigma) = \pm \frac{\alpha^i}{2\sigma}, \quad (9)$$

where the α^i are an $N \times N$ representation of the $SU(2)$ algebra: $[\alpha^i, \alpha^j] = 2i\epsilon^{ijk}\alpha^k$. The Casimir C for this algebra is defined by $\sum (\alpha^i)^2 = C \mathbf{1}_N$, where $\mathbf{1}_N$ the $N \times N$ identity matrix. We will focus on the irreducible $N \times N$ representation for which $C = N^2 - 1$. This noncommutative scalar field configuration (9) describes a fuzzy two-sphere[17] with a physical radius

$$R(\sigma) = \lambda \sqrt{\text{Tr}[\Phi^i(\sigma)^2]/N} = \frac{\sqrt{C}\pi\ell_s^2}{\sigma} = \frac{N\pi\ell_s^2}{\sigma} \sqrt{1 - 1/N^2}. \quad (10)$$

Hence the solution describes a “fuzzy funnel” in which the D-strings expand to span the $X^{1,2,3}$ hyperplane at $\sigma = 0$. This geometry can be compared to the D3-brane solution (4) after relabeling $\sigma \rightarrow \lambda\phi$ and $R \rightarrow r$. We see that both descriptions yield the same geometry in the limit of large N , up to $1/N^2$ corrections. There is similar agreement for other quantities. The energy can be evaluated from the boundary term in eqn. (7) and is found to be,

$$E = NT_1 \int_0^\infty d\sigma + (1 - 1/N^2)^{-1/2} T_3 \int 4\pi R^2 dR, \quad (11)$$

while the non-abelian Wess-Zumino couplings of ref. [13] include,

$$i\lambda\mu_1 \int \text{Tr} P[\mathbf{i}_\Phi \mathbf{i}_\Phi C^{(4)}] = \mp i\mu_3 (1 - 1/N^2)^{-1/2} \int dt 4\pi R^2 dR C_{t123}^{(4)} \quad (12)$$

which indicates that, upto $1/N^2$ corrections, the funnel is a source for precisely the correct RR four-form field to be identified with a D3-brane. Hence at least in the large N limit, the two dual descriptions are in good agreement.

Note, however, that the Born-Infeld actions, eqs. (1) and (6), only describe low energy dynamics and do not provide a complete theory in either case. Therefore, in general, one must regard these two dual descriptions as complimentary. That is, the D-string theory gives reliable results for small r near the core of the spike, while the D3-brane theory is valid for r large. However, this makes the strong agreement between the two dual descriptions seem remarkable, so one must be more precise about the range of validity of each theory[4]. The Born-Infeld action does not include higher derivative stringy corrections, but one finds that these are negligible for the D3-brane bion when $R \gg \ell_s$ and for the D-string funnel when $R \ll N\ell_s$. The nonabelian D-string action (6) also neglects certain higher commutator corrections[18], and one can argue that their contributions remain small for $R \ll \sqrt{N}\ell_s$. Hence for large N , there is a large interval in which both low energy actions give a reliable account of the $D3 \perp D1$ -intersections. This overlap is the source of the strong agreement between the two descriptions in this large N regime.

3 Dual Formulations of D5⊥D1 Intersections

Given that the fuzzy geometry encountered in eq. (9) was put in by hand it is natural to consider generalizations involving fuzzy geometries other than the two-sphere. In this section, following ref. [5], we will consider the D-string theory in which five transverse scalars are excited and hence extend the above analysis by considering fuzzy four-spheres.

Our starting point is again the low energy action for N D-strings, however, we now consider static configurations involving five (rather than three) nontrivial scalars, Φ^i with $i = 1, \dots, 5$. In this case, the action (6) becomes

$$S = -T_1 \int d^2\sigma \text{STr} \left\{ 1 + \lambda^2 (\partial_\sigma \Phi^i)^2 + 2\lambda^2 \Phi^{ij} \Phi^{ji} + 2\lambda^4 (\Phi^{ij} \Phi^{ji})^2 - 4\lambda^4 \Phi^{ij} \Phi^{jk} \Phi^{kl} \Phi^{li} + \right. \\ \left. + 2\lambda^4 (\partial_\sigma \Phi^i)^2 \Phi^{jk} \Phi^{kj} - 4\lambda^4 \partial_\sigma \Phi^i \Phi^{ij} \Phi^{jk} \partial_\sigma \Phi^k + \frac{\lambda^6}{4} (\epsilon^{ijklm} \partial_\sigma \Phi^i \Phi^{jk} \Phi^{lm})^2 \right\}^{1/2}, \quad (13)$$

where we have introduced the convenient notation, $\Phi^{ij} \equiv \frac{1}{2} [\Phi^i, \Phi^j]$.

To construct a new funnel solution, we consider the following ansatz:

$$\Phi^i(\sigma) = \pm \frac{R(\sigma)}{\sqrt{c\lambda}} G^i, \quad i = 1, \dots, 5, \quad (14)$$

where $R(\sigma)$ is the radial profile and G^i are the matrices constructed in ref. [19] — see also ref. [20]. The definition and many useful properties of the G^i matrices may be found in refs. [19, 5]. Here, we simply note that the G^i are given by the totally symmetric n -fold tensor product of 4×4 gamma matrices, and that the dimension of the matrices is related to the integer n by

$$N = \frac{(n+1)(n+2)(n+3)}{6}. \quad (15)$$

The solution will describe a funnel whose cross-section is a fuzzy four-sphere with a physical radius $R(\sigma) = \lambda \sqrt{\text{Tr}[\Phi^i(\sigma)^2]/N}$. The latter identification requires choosing the normalization constant c to be the “Casimir” associated with the G^i matrices, *i.e.*, $G^i G^i = c \mathbf{1}_N$, which is given by $c = n(n+4)$.

An immediate puzzle is that to leading order in λ the equations of motion for this system are exactly the same as for the case of three scalars in the previous section and therefore the funnel profile will again be $R \sim \sigma^{-1}$. In fact, this behavior is universal to any D-string funnel in the small R regime[4]. If however the present configuration is to represent a D5-brane then one would anticipate that the funnel would follow a profile given by $R \sim \sigma^{-1/3}$ appropriate for a harmonic function in five spatial directions. As we now show the resolution of this puzzle is that the higher order terms appearing in the action (13) effect the necessary transition.

Since dealing with the full equations of motion following from this action is tedious and largely unenlightening we will not consider them here. We do note however that inserting the

ansatz (14) into these equations yields a single differential equation for $R(\sigma)$. Knowing this we simply insert our ansatz into the action and making use of the identities satisfied by the G^i obtain an action for the radial profile,

$$S = -NT_1 \int d^2\sigma \sqrt{1 + (R')^2} [1 + 4R^4/(c\lambda^2)], \quad (16)$$

which then in analogy with eqs. (2) and (7) yields the following bound on the energy,

$$E \geq NT_1 \int d\sigma \left(1 \pm R' \sqrt{8R^4/(c\lambda^2) + 16R^8/(c\lambda^2)^2} \right). \quad (17)$$

This is again a sum of the trivial term and a topological term. The equality is saturated when

$$R' = \mp \sqrt{8R^4/(c\lambda^2) + 16R^8/(c\lambda^2)^2}. \quad (18)$$

Note that this equation is also compatible with the full equation of motion. We may write the solutions of eq. (18) in terms of elliptic functions, however, the geometry is more clearly exhibited by considering various limits. For small R , the R^4 term under the square root dominates, and we find the funnel solution

$$R(\sigma) \simeq \frac{\sqrt{c}\lambda}{2\sqrt{2}\sigma}. \quad (19)$$

This is precisely the leading order solution found above with the universal behavior: $R \sim \sigma^{-1}$. However, for large R the equation becomes $R' = \mp 4R^4/(c\lambda^2)$, with the solution

$$R(\sigma) \simeq \left(\frac{12\sigma}{c\lambda^2} \right)^{-1/3}, \quad (20)$$

which is precisely the harmonic behavior that we anticipated for a D5-brane to appear at $\sigma = 0$. The cross-over between the universal and harmonic expansion of the funnel occurs when the two terms under the square root are comparable, *i.e.*,

$$R_c \sim (c\lambda^2/2)^{1/4} = (2\pi^2 c)^{1/4} \ell_s. \quad (21)$$

Note that for large c (and hence large N), this is a macroscopic distance scale, *i.e.*, $R_c \gg \ell_s$.

If this configuration indeed corresponds to a D5-brane then the fuzzy funnel must act as the source for correct RR fields. One can easily show that the following term in the non-abelian Wess-Zumino action[13, 21] gives,

$$\begin{aligned} -\frac{\lambda^2 \mu_1}{2} \int \text{STr } P [(i_\Phi i_\Phi)^2 C^{(6)}] &= -\frac{\lambda^3 \mu_1}{2} \int d\sigma d\tau C_{012345}^{(6)} \text{STr} (\epsilon^{ijklm} \Phi^i \Phi^j \Phi^k \Phi^l \partial_\sigma \Phi^m) \\ &= \pm \frac{6N(n+2)}{c^{3/2}} \mu_5 \int d\tau dR \Omega_4 R^4 C_{012345}^{(6)}, \end{aligned} \quad (22)$$

where $\mu_5 = \mu_1/(2\pi\ell_s)^4$ and $\Omega_4 = 8\pi^2/3$ is the area of a unit four-sphere. This is precisely the D5-source term we expect. Further, the number of D5-branes is given by

$$\frac{6N(n+2)}{c^{3/2}} = \frac{(n+1)(n+2)^2(n+3)}{n^{3/2}(n+4)^{3/2}} \simeq n, \quad (23)$$

for large N . Thus, in this limit, the funnel appears to expand into n D5-branes. One may also calculate the energy of this system to be,

$$\begin{aligned} E &= NT_1 \int_0^\infty d\sigma [1 + 4R^4/(c\lambda^2)]^2 \\ &= NT_1 \int_0^\infty d\sigma + \frac{6N}{c} T_5 \int_0^\infty \Omega_4 R^4 dR + NT_1 \int_0^\infty dR - \Delta E, \end{aligned} \quad (24)$$

where $T_5 = T_1/(2\pi\ell_s)^4$. Where the first term corresponds to the energy of the N semi-infinite D-strings that we started with and the second (in the large N limit) to n orthogonal D5-branes spanning the $X^{1,2,3,4,5}$ hyperplane. As this is not a supersymmetric configuration there are other contributions to the energy. The third term in eqn. (24) seems to correspond to N semi-infinite D-strings extending radially *inside* the D5-brane. The fourth term is a finite binding energy which can be evaluated numerically to be: $\Delta E \approx 1.0102Nc^{1/4}T_1\ell_s$.

We now turn to a discussion of this new brane intersection in terms the world volume theory of the D5-branes. The analysis of refs. [10, 22] indicates that N D-strings may terminate on a collection of n D5-branes so long as they act as a source for second Chern class, *i.e.*, $\int_{S^4} \text{Tr} F \wedge F = 8\pi^2 N$ for any S^4 surrounding the end point of the D-strings. We note that in this case producing a smooth resolution of the D-string endpoint requires a non-abelian gauge field configuration and therefore $n > 1$. The action for n D5-branes reduces to

$$S_5 = -T_5 \int d^6\sigma \text{STr} \sqrt{-\det(G_{ab} + \lambda^2 \partial_a \phi \partial_b \phi + \lambda F_{ab})} \quad (25)$$

if we excite only one $U(1)$ scalar, as well as the nonabelian gauge field. In order to construct a D-string spike solution to this theory we proceed in analogy to the bion spike in the D3-brane theory. We work in spherical polar coordinates in the D5-brane world volume with a radius r and angles $\alpha^i, i = 1, \dots, 4$ with metric,

$$ds^2 = G_{ab} d\sigma^a d\sigma^b = -dt^2 + dr^2 + r^2 g_{ij} d\alpha^i d\alpha^j, \quad (26)$$

where g_{ij} is the metric on a four-sphere with unit radius,

$$g_{ij} = \text{diag}[1, \sin^2(\alpha^1), \sin^2(\alpha^1) \sin^2(\alpha^2), \sin^2(\alpha^1) \sin^2(\alpha^2) \sin^2(\alpha^3)] . \quad (27)$$

Now we look for bion spike solutions with a “nearly spherically symmetric” ansatz: The scalar is only a function of the radius, *i.e.*, $\phi = \phi(r)$. For the gauge field, we require that $A_r = 0$ while the angular components are independent of r , *i.e.*, $A_{\alpha^i} = A_{\alpha^i}(\alpha^j)$. Examining the full equations of motion shows that this is a consistent ansatz. For the non-vanishing components of the field strength, we introduce the convenient notation: $\tilde{F}_{ij} \equiv \lambda F_{\alpha^i \alpha^j}$.

The gauge field equation of motion can be written

$$D_i \left[\sqrt{g} \frac{r^4 \tilde{F}^{ij} + \frac{1}{4} {}^* \tilde{F}^{ij} \tilde{F}_{kl} {}^* \tilde{F}^{kl}}{\sqrt{r^8 + \frac{1}{2} r^4 \tilde{F}_{ij} \tilde{F}^{ij} + \frac{1}{16} (\tilde{F}_{ij} {}^* \tilde{F}^{ij})^2}} \right] = 0, \quad (28)$$

It is easy to see that if we choose a self-dual field strength $\tilde{F}_{ij} = {}^* \tilde{F}_{ij}$, the above equation reduces to the usual Yang-Mills equations, which are then automatically satisfied due to the Bianchi identity. Further the scalar equation of motion implies that $\tilde{F}_{ij} {}^* \tilde{F}^{ij}$ be independent of the angles, *i.e.*, we are considering homogeneous instantons on the four-sphere. The scalar equation of motion reduces to,

$$\lambda^2 (\phi')^2 = \frac{1}{\tilde{B}^2 (\frac{nr^4}{\lambda^2} + \frac{3}{2} N)^2 - 1}. \quad (29)$$

where \tilde{B} is a dimensionless integration constant. To compare with the radial profile (18) found in the D-string description, we identify the physical transverse distance as $\sigma = \lambda \phi$, and equate the radii $r = R$. Then we see the form of the two equations agrees provided that we set $\tilde{B} = 2/3N$. Complete agreement of the equations requires the coefficients $4/c$ and $2n/(3N)$ to be equal, and in fact, this equality is achieved in the large N limit. Hence in this limit we have complete agreement for the geometry determined by the two dual approaches!

For the spherically symmetric spike describing semi-infinite D-strings, the energy is easily evaluated to be

$$\begin{aligned} E &= T_5 \int \sqrt{g} d^4 \alpha dr \sqrt{1 + \lambda^2 (\phi')^2} (nr^4 + \frac{3}{2} N \lambda^2) \\ &= NT_1 \int d\sigma + nT_5 \int \Omega_4 r^4 dr + NT_1 \int dr - \Delta E, \end{aligned} \quad (30)$$

with $\Delta E \approx 1.0102N(6N/n)^{1/4}T_1\ell_s$. Again we have full agreement with the D-string result (24) in the limit of large N . Note that in analogy with the D3 \perp D1 analysis, the above result includes a contribution of precisely n D5-branes, while the D-string calculations yield $1/n$ corrections to the coefficient of this term.

The one point we have not yet addressed is finding an actual gauge field solution corresponding to a homogeneous instanton on the four-sphere. The ADHM construction reduces the problem of finding instanton configurations on \mathbb{R}^4 to an explicit algebraic procedure — see, *e.g.*, ref. [23]. Due to conformal invariance of the Yang-Mills system, instanton solutions on the four-sphere can then be produced with the usual stereographic projection of S^4 onto \mathbb{R}^4 . For instanton number $N = 1$ (or -1) and gauge group $SU(2)$, one finds that the unit size instanton located at the origin of \mathbb{R}^4 projects to a homogeneous instanton configuration on the four-sphere. This configuration is described in some detail in ref. [5] for the interested reader. Replacing the fundamental $SU(2)$ generators by an $n \times n$ representation allows us to embed

this homogeneous solution in an $SU(n)$ gauge theory. The instanton number of the resulting $SU(n)$ gauge field is maximized by choosing the irreducible representation, giving

$$N = \frac{1}{8\pi^2} \int_{S^4} \text{Tr } F \wedge F = \frac{n(n^2 - 1)}{6}. \quad (31)$$

Now a key result is that using a theorem by H.C. Wang[24], one may prove that this is the *homogeneous* instanton configuration on the four-sphere with the largest possible value of the second Chern class — see discussion in ref. [5]. Note that for large n , the upper bound (31) on N agrees exactly with the mysterious restriction (15) that appears in the construction of the fuzzy funnel! While this agreement is a remarkable success of the duality between the D-string and D5-brane descriptions, it appears that this restriction arises from our use of a spherically symmetric ansatz. Presumably the restriction can be evaded by considering more general field configurations, although as a practical matter the analysis becomes much more difficult.

In this section we have examined the orthogonal intersection of a collection of N D1-branes ending on a set of n D5-branes. We have done so within the world volume theories of both the D1-branes and the D5-branes. The two constructions agree exactly in the large N limit and provide dual, complementary pictures of this configuration. As before, one can verify that this agreement arises in the large N regime because there is a large interval, $\ell_s^2 \ll r^2 \ll n\ell_s^2$, in which both theories give reliable results. In analogy with the $D3 \perp D1$ -system in the previous section, the theory on the D1-branes involving the fuzzy four-sphere is valid far out along the spike while near the D5-branes the configuration is best described by the instantonic configuration in the D5-brane world volume theory.

4 Discussion

We have discussed two dual descriptions of both the $D3 \perp D1$ - and the $D5 \perp D1$ -systems. At least in a large N regime, the dual theories yield good agreement for the energy, the RR couplings and the geometry of brane configurations. While one may regard the two different types of D-branes as giving dual descriptions of the same intersection, we stress that our analysis has been limited to the *low energy* world-volume theory in each case. Hence in neither case do we have a complete description as in both theories the configurations of interest contain singularities which are not resolved by the low energy action. Our analysis therefore yields two complimentary descriptions of the $D3 \perp D1$ - and $D5 \perp D1$ -intersections. In the region far away from the D3- or D5-branes, the fuzzy funnel construction of the D1-brane theory is reliable, while nearby, the bion picture of the D3- or D5-brane theory is trustworthy. Still a careful analysis[4, 5] shows that in the large N regime, the low energy descriptions of the two dual theories are both reliable over a large interval of the intersection geometry. This overlap can then be understood as the source of the strong agreement in this regime.

One can go beyond the static configurations to analyse small perturbations propagating in these brane intersections[6, 7, 4, 5]. However, once again, one must be careful to reconsider the

regime in which these linearized perturbation solutions are valid. In particular, ref. [7] claims there is no mechanism to suppress the propagation of high angular momentum modes at the bion core, but this conclusion is not valid since their linearized analysis breaks down far from the core for these modes[4]. Further the dual D1-brane description involving a fuzzy funnel makes clear that there are only N^2 angular momentum modes in the low energy spectrum. (See ref. [4] for the full details.) While only a partial analysis of the linearized fluctuations for the $D5 \perp D1$ -intersection has been made[5], it seems that the fuzzy four-sphere captures oscillations of both the geometry and the internal instantonic degrees of freedom.

There are many other extensions of the present constructions that might be considered. For instance, one might construct fuzzy funnels with other noncommutative geometries such as those appearing in ref. [25]. For example, the coset $SU(3)/U(2)$ would seem to yield a brane intersection of D-strings with D5-branes with a more exotic five-brane geometry. Another interesting intersection is the $D7 \perp D1$ -system. In this case, the configuration is supersymmetric and so the world-volume field theory configurations may be more reliable. In the D7-brane theory, one would have to consider gauge field configurations on the six-sphere with nonvanishing third Chern character. The dual D-string funnel would now involve a fuzzy S^6 .

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